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Quantum mechanics as non-commutative symplectic geometry

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Abstract. We construct consistent differential calculi on the algebra generated by operators satisfying the canonical commutation relation. This leads to a mathematical framework in which quantum mechanics can be understood as (non-commutative) symplectic geometry. We point out the possibility to describe interactions 'geometrically' as deformations of the differential calculus.

1. Introduction

Recently there has been considerable interest in non-commutative geometry [1] as a framework for physical theories [2–5] and as a tool for studying certain structures which appear in some physical models [6–8]. The latter in particular concerns quantum groups for which non-commutative differential geometry has been developed by Woronowicz [9], and Wess and Zumino [10] (see also [11] and the references cited there). The present work is influenced by the latter. We address the question of the role which differential calculi on operator algebras could play in physics. We demonstrate that the classical notions of symplectic geometry can be transferred to the algebra generated by operators satisfying the canonical commutation relations. Quantum mechanics can then be described in terms of non-commutative symplectic geometry.

Similar ideas appeared in the work of Dubois-Violette *et al* [3]. In contrast to our approach they consider the derivations of an algebra as the analogue of the classical notion of vector fields and take them as the starting point to develop concepts of non-commutative geometry (concentrating on the algebra of complex $n \times n$ matrices). Thereby they overtake the rules of the classical differential calculus.

Also the algebra considered in the present work is consistent with ordinary differential calculus. However, the apparent uniqueness of classical calculus of differential forms (built on the commutative algebra of functions on a manifold) disappears in the non-commutative regime. There are algebras which are only consistent with deformed differential calculi. This opens a variety of general questions some of which we discuss in the context of our example. There may be deformation parameters (or functions) in a differential calculus which are 'inessential' in the sense that they can be transformed away by an automorphism of the algebra. Can we assign a physical role to 'essential' deformations?

Section 2 introduces differential calculus on the 'Heisenberg algebra' $\mathcal{A}$ generated by operators satisfying the canonical commutation relation. We confine our considerations to one pair of conjugate variables. The framework easily extends to the general case, but the problem of solving the consistency conditions will become more complicated. In section 2 we study this problem for the one-dimensional case. The appendix contains the proof of lemma 2.1 in section 2. The commutation relations of the resulting consistent differential calculi are summarized in section 3 and their relation with differential calculus on the 'quantum plane' [12, 10] is discussed.

Vector fields and flows on $\mathcal{A}$, and an inner product between vector fields and differential forms are introduced in sections 4 and 5, respectively. In section 6 we use these tools to formulate symplectic geometry and symplectic dynamics on the operator algebra. Section 7 contains some conclusions.

2. Differential calculus on the operator algebra

In this section we first introduce the Heisenberg algebra $\mathcal{A}$ in a form convenient for the subsequent calculations. We then recall the notion of a differential calculus on an algebra. The rest of this section deals with the problem of determining consistent differential calculi on $\mathcal{A}$. Our main result is summarized in theorem 2.1 at the end of this section.

We will restrict our considerations to a one-dimensional system with 'position' operator q and 'momentum' operator p satisfying the canonical commutation relation

$$[q, p] = \mathrm{i}\hbar \mathbb{1}. \tag{2.1}$$

In terms of

$$\xi^1 = (\xi^1)^\dagger = q \qquad \xi^2 = (\xi^2)^\dagger = p \tag{2.2}$$

this takes the form

$$\epsilon_{ij}\xi^i\xi^j = \mathrm{i}\hbar\mathbb{1} \tag{2.3}$$

where

$$\epsilon = (\epsilon_{ij}) = (\epsilon^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4}$$

Using the permutation operator

$$P^{ij}_{k\ell} = \delta^i_\ell \delta^j_k \tag{2.5}$$

and

$$(\mathbf{1} - P)^{ij}_{k\ell} = \delta^i_k\delta^j_\ell - \delta^i_\ell\delta^j_k = \epsilon^{ij}\epsilon_{k\ell} \tag{2.6}$$

a convenient index-free formulation of (2.1) is

$$(\mathbf{1} - P)\xi\xi = \mathrm{i}\hbar\epsilon. \tag{2.7}$$

Let $\mathcal{A}$ denote the $\mathbb{C}$-algebra of polynomials generated by ξ^i and $\mathbf{1}$. With the ξ^i we formally associate 'differentials' $\mathrm{d}\xi^i$ which generate the space $\Lambda^1(\mathcal{A})$ of 1-forms as an $\mathcal{A}$-bimodule. We also set $\mathrm{d}\mathbf{1} = 0$.

A differential calculus [9] on $\mathcal{A}$ is a prescription $\wedge$ of how to multiply 1-forms (and then also r-forms) together with a ($\mathbb{C}$-linear) 'exterior derivative' operator d such that†

$$\mathrm{d}^2 = 0 \tag{2.8}$$

$$\mathrm{d}(\omega \wedge \omega') = (\mathrm{d}\omega) \wedge \omega' + (-1)^r \omega \wedge \mathrm{d}\omega' \tag{2.9}$$

$$(\omega \wedge \omega')^\dagger = (-1)^{rr'} \omega'^\dagger \wedge \omega^\dagger \tag{2.10}$$

$$(\mathrm{d}f)^\dagger = \mathrm{d}(f^\dagger) \tag{2.11}$$

where ω and ω' are r- and r'-forms, respectively, and $f \in \mathcal{A}$.

We will assume commutation relations between algebra elements and differentials of the form

$$f\mathrm{d}\xi^i = \mathrm{d}\xi^k\, \Theta(f)^i_k \qquad (\forall f \in \mathcal{A}) \tag{2.12}$$

with $\Theta(f)^i_k \in \mathcal{A}$. This enforces

$$\Theta(fh)^i_j = \Theta(f)^k_j \Theta(h)^i_k \qquad (\forall f, h \in \mathcal{A}) \tag{2.13}$$

$$\Theta(\mathbf{1})^i_j = \delta^i_j. \tag{2.14}$$

To specify a differential calculus means to specify the operator Θ which is therefore our central object. The problem is to find all *consistent* differential calculi on $\mathcal{A}$. The consistency equations which have to be satisfied arise from the following procedures (see [9] in the case of quantum groups):

(C1) Apply d to the canonical commutation relation (2.3) and use (2.12) to commute all differentials to the left.

(C2) Apply the conjugation † to (2.12).

(C3) Commute the differentials $\mathrm{d}\xi^i$ through the canonical commutation relation (2.3).

(C4) Acting with d on (2.12) (with ξ^j instead of f) leads to commutation relations for the differentials. Commute ξ^i through these relations.

We were unable to solve this problem in full generality. In the following we will make the *additional assumption*‡ that $\Theta(\xi^i)$ is *linear* in the ξ^j, i.e.

$$\Theta(\xi^i)^j_k = \Theta^{ij}_{\ell k}\xi^\ell \tag{2.15}$$

so that

$$\xi^i \mathrm{d}\xi^j = \Theta^{ij}_{\ell k} \mathrm{d}\xi^k \xi^\ell. \tag{2.16}$$

† Here we have chosen the definition used by Woronowicz [9]. (2.10) deviates from Connes' prescription. He defines $(a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n)^\dagger = \mathrm{d}(a_n^\dagger) \ldots \mathrm{d}(a_1^\dagger) a_0^\dagger\ \forall a_i \in \mathcal{A}$.

‡ This is motivated by the form of equation (2.20).

It is convenient to write this in the compact form

$$\xi \mathrm{d}\xi = \hat{\Theta} \mathrm{d}\xi \, \xi \tag{2.17}$$

where

$$\hat{\Theta} = \Theta P. \tag{2.18}$$

Let us first evaluate the consistency condition (C1). Acting with d on (2.3) gives

$$\epsilon_{ij}(\mathrm{d}\xi^i \, \xi^j + \xi^i \, \mathrm{d}\xi^j) = 0 \tag{2.19}$$

which together with (2.12) implies

$$\epsilon_{ij} \Theta(\xi^i)^j_k + \epsilon_{kj}\xi^j = 0. \tag{2.20}$$

Using our compact notation, (2.20) with (2.15) becomes

$$(\mathbf{1} - P)(\mathbf{1} + \hat{\Theta}) = 0 \tag{2.21}$$

and therefore

$$(\mathbf{1} - P)(\mathbf{1} - \Theta) = 0. \tag{2.22}$$

This equation expresses the first consistency condition.

The second consistency condition can be expressed as

$$\bar{\Theta}\Theta = \mathbf{1} \tag{2.23}$$

where the bar denotes complex conjugation. This is seen as follows:

$$\begin{aligned} P\mathrm{d}\xi \, \xi &= (\xi \, \mathrm{d}\xi)^\dagger = (\Theta P \mathrm{d}\xi \, \xi)^\dagger \\ &= \bar{\Theta} P P \xi \, \mathrm{d}\xi = \bar{\Theta} \xi \, \mathrm{d}\xi = \bar{\Theta}\Theta P \mathrm{d}\xi \, \xi. \end{aligned}$$

Let us now turn to the third consistency condition. In order to formulate it in a convenient and compact way we have to introduce some more notation. Θ may be viewed as a 2×2 matrix where the entries are 2×2 matrices themselves:

$$(\Theta^i_j)^k_\ell = \Theta^{ik}_{j\ell} \qquad \Theta = (\Theta^i_j) \tag{2.24}$$

(i.e. Θ is a sum of tensor products of two 2×2 matrices). We define

$$\mathrm{Tr}_1 \, \Theta = \Theta^i_i \tag{2.25}$$

i.e.

$$(\mathrm{Tr}_1 \, \Theta)^k_\ell = \Theta^{ik}_{i\ell}. \tag{2.26}$$

Using (2.22) and (2.23) the condition (C3) can be expressed as follows. The proof is given in the appendix.

Lemma 2.1.

$$\mathrm{Re}\,\Theta = \mathbf{1} \qquad \mathrm{Tr}_1(\Theta - \mathbf{1}) = 0\,. \tag{2.27}$$

□

It is now convenient to write

$$\Theta = \mathbf{1} + \mathrm{i}\Omega. \tag{2.28}$$

Then (2.22), (2.23) and (2.27) are converted into

$$\bar{\Omega} = \Omega \tag{2.29}$$

$$P\Omega = \Omega \tag{2.30}$$

$$\mathrm{Tr}_1\,\Omega = 0 \tag{2.31}$$

$$\Omega^2 = 0. \tag{2.32}$$

These four equations express the consistency conditions (C1)–(C3) (under the linearity assumption imposed on Θ).

An interesting question is how a differential calculus (specified by a matrix Θ, respectively Ω) transforms if we perform a transformation of the algebra generators p and q which preserves the canonical commutation relation and the Hermiticity. Restricting to only linear transformations, the corresponding transformation group is $\mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(2)$. The following result may then come as no surprise.

Lemma 2.2. (2.29)–(2.32) are invariant under transformations

$$\Omega \mapsto (S \otimes S)\Omega(S^{-1} \otimes S^{-1}) \tag{2.33}$$

with $S \in \mathrm{SL}(2,\mathbb{R})$.

Proof. (2.29) and (2.32) are evidently invariant. The invariance of (2.30) is a consequence of the identity

$$(\mathbf{1} - P)S \otimes S = \mathbf{1} - P.$$

In order to demonstrate the invariance of (2.31) we write Ω as

$$\Omega = \sum_a X_a \otimes Y_a$$

with 2×2 matrices X_a and Y_a. Then

$$\begin{aligned}\mathrm{Tr}_1[(S \otimes S)\Omega(S^{-1} \otimes S^{-1})] &= \sum_a \mathrm{Tr}_1[SX_aS^{-1} \otimes SY_aS^{-1}]\\ &= \sum_a \mathrm{Tr}(SX_aS^{-1})SY_aS^{-1}\\ &= \sum_a (\mathrm{Tr}\,X_a)SY_aS^{-1}\end{aligned}$$

which vanishes if $\mathrm{Tr}_1\,\Omega = 0$. □

The significance of this result lies in the fact that we may regard differential calculi as identical, if they are related by an $SL(2,\mathbb{R})$-transformation. This paves the way towards the general solution of the above equations for Ω. (2.29) and (2.31) are solved if we set

$$\Omega = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2.34}$$

with real 2×2 matrices A, B, C. (2.32) now leads to

$$\begin{aligned} AB &= BA \\ AC &= CA \\ BC &= CB \end{aligned} \tag{2.35}$$

and

$$A^2 + BC = 0. \tag{2.36}$$

The algebra of real 2×2 matrices has at most two-dimensional Abelian subalgebras and these are spanned by $\mathbf{1}_{(2)}$ (the 2×2 unit matrix) and a tracefree matrix T. (2.35) then implies

$$\begin{aligned} A &= \kappa \mathbf{1}_{(2)} + c_1 T \\ B &= \mu \mathbf{1}_{(2)} + c_2 T \\ C &= \nu \mathbf{1}_{(2)} + c_3 T \end{aligned} \tag{2.37}$$

with real coefficients κ, μ, ν and c_K. (2.35) is then satisfied and we are left with (2.36) and (2.30).

Due to lemma 2.2 it is sufficient to solve these equations for representatives from each $SL(2,\mathbb{R})$ orbit of trace-free matrices. Besides $T = 0$ there are three families of orbits, representatives of which are considered in the following. The orbit $T = 0$ is included as the limit $c_K \to 0$.

(a) Solving (2.30) with

$$T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.38}$$

reduces Ω to the form

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \nu & 0 & 0 & 0 \\ \nu & 0 & 0 & 0 \\ c & \nu & -\nu & 0 \end{pmatrix} \tag{2.39}$$

and (2.36) is automatically satisfied.

(b) Choosing

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.40}$$

(2.30) leads to

$$\Omega = \begin{pmatrix} 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2\nu & 0 & 0 \end{pmatrix} \tag{2.41}$$

and (2.36) is again automatically satisfied.

(c) The last family of orbits is represented by

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.42}$$

and (2.30) yields

$$\Omega = \begin{pmatrix} \kappa & -\mu & \mu & \kappa \\ \mu & \kappa & -\kappa & \mu \\ \mu & \kappa & -\kappa & \mu \\ -\kappa & \mu & -\mu & -\kappa \end{pmatrix}. \tag{2.43}$$

Again, (2.36) is satisfied.

The three solutions for Ω define consistent first-order differential calculi. Further restrictions arise from the extension to higher-order forms with a consistent $\wedge$ product. This is the consistency condition (C4) which we have to work out next.

Acting with the exterior derivative operator on (2.17) yields

$$(\mathbf{1} + \hat{\Theta})\mathrm{d}\xi \wedge \mathrm{d}\xi = 0. \tag{2.44}$$

In the following we will often drop the wedge symbol between 1-forms. Using (2.28) the last equation is turned into

$$(\mathbf{1} + P + \mathrm{i}\Omega)P\,\mathrm{d}\xi\,\mathrm{d}\xi = 0 \tag{2.45}$$

which together with (2.32) implies

$$-\tfrac{1}{2}\mathrm{i}\Omega(\mathbf{1} + P)P\,\mathrm{d}\xi\,\mathrm{d}\xi = 0. \tag{2.46}$$

Added to the previous equation this leads to

$$(\mathbf{1} + \hat{R})\,\mathrm{d}\xi\,\mathrm{d}\xi = 0. \tag{2.47}$$

Here we have introduced

$$\hat{R} = RP \qquad R = \mathbf{1} + \tfrac{1}{2}\mathrm{i}\Omega(\mathbf{1} - P) \tag{2.48}$$

which has the property

$$\hat{R}^2 = \mathbf{1} \tag{2.49}$$

as a consequence of (2.30). An equation of the form $A\,\mathrm{d}\xi\,\mathrm{d}\xi = 0$ is then equivalent to $A(\mathbf{1} - \hat{R}) = 0$. Using (2.17) and (2.47) the consistency condition (C4) can now be formulated as†

$$\begin{aligned} 0 &= (\mathbf{1} + \hat{R})_{23}\xi\,\mathrm{d}\xi\,\mathrm{d}\xi \\ &= (\mathbf{1} + \hat{R})_{23}\hat{\Theta}_{12}\,\mathrm{d}\xi\,\xi\,\mathrm{d}\xi \\ &= (\mathbf{1} + \hat{R})_{23}\hat{\Theta}_{12}\hat{\Theta}_{23}\,\mathrm{d}\xi\,\mathrm{d}\xi\,\xi \end{aligned}$$

which is equivalent to

$$(\mathbf{1} + \hat{R})_{23}\hat{\Theta}_{12}\hat{\Theta}_{23}(\mathbf{1} - \hat{R})_{12} = 0. \tag{2.50}$$

Evaluating this equation with (2.39), (2.41) and (2.43), respectively, we find that only (2.39) survives with non-zero entries. In the latter case (2.50) requires $\nu = 0$, but does not restrict the parameter $c \in \mathbb{R}$.

Let us summarize the results of this section.

Theorem 2.1. All consistent differential calculi on the Heisenberg algebra $\mathcal{A}$ satisfying the linearity condition (2.15) are given modulo $\mathrm{SL}(2,\mathbb{R})$ transformations by

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{i}c & 0 & 0 & 1 \end{pmatrix}$$

with $c \in \mathbb{R}$. □

For this class of differential calculi equation (2.47) becomes

$$(\mathbf{1} + P)\,\mathrm{d}\xi\,\mathrm{d}\xi = 0 \tag{2.51}$$

which means that the wedge product is the classical one.

3. Consistent differential calculi and transformations

In this section we further investigate the class of consistent differential calculi found in the previous section. The most important result is that the deformation parameter c can be transformed away by an automorphism of the algebra $\mathcal{A}$ (see theorem 3.1). The remainder of this section contains some more technical remarks and briefly discusses the relation of our work with differential calculus on the 'quantum plane'.

For the class of consistent differential calculi on $\mathcal{A}$ which we obtained in the previous section, the commutation relations between p and q and their differentials are (modulo $\mathrm{SL}(2,\mathbb{R})$ transformations)

$$\begin{aligned} q\,\mathrm{d}q &= \mathrm{d}q\,q \qquad & p\,\mathrm{d}q &= \mathrm{d}q\,p \\ q\,\mathrm{d}p &= \mathrm{d}p\,q \qquad & p\,\mathrm{d}p &= \mathrm{d}p\,p + \mathrm{i}c\,\mathrm{d}q\,q \end{aligned} \tag{3.1}$$

and we have the classical $\wedge$ product

$$\mathrm{d}q \wedge \mathrm{d}q = 0 \qquad \mathrm{d}q \wedge \mathrm{d}p = -\mathrm{d}p \wedge \mathrm{d}q \qquad \mathrm{d}p \wedge \mathrm{d}p = 0. \tag{3.2}$$

† The indices indicate the action on the respective components of the threefold tensor product.

Remark. Motivated by corresponding results in the case of quantum groups [9], we look for a representation of d in the form†

$$\mathrm{d}f = [\rho, f]_- \qquad (\forall f \in \mathcal{A}) \tag{3.3}$$

$$\mathrm{d}\theta = [\rho, \theta]_+ \qquad (\forall \theta \in \Lambda^1(\mathcal{A})) \tag{3.4}$$

with a 1-form ρ. The Leibniz rule is then guaranteed. Inserting the ansatz $\rho = g\,\mathrm{d}q + h\,\mathrm{d}p$ with $g, h \in \mathcal{A}$ in (3.3) with $f = q$ and $f = p$, respectively, we find

$$\rho = \frac{1}{\mathrm{i}\hbar}\left(\left(\frac{c}{3\hbar}q^3 - p\right)\mathrm{d}q + q\,\mathrm{d}p\right) \tag{3.5}$$

for the differential calculus given by (3.1) and (3.2). Furthermore,

$$\rho \wedge \rho = \frac{1}{\mathrm{i}\hbar}\,\mathrm{d}q \wedge \mathrm{d}p \tag{3.6}$$

which commutes with all $f \in \mathcal{A}$ so that

$$\mathrm{d}^2 f = [\rho, [\rho, f]_-]_+ = [\rho \wedge \rho, f]_- = 0 \tag{3.7}$$

as required for an exterior derivative. □

As a consequence of the commutation relations (3.1) we have

$$p^n\,\mathrm{d}p = \mathrm{d}p\,p^n + \mathrm{d}q\,c[\tfrac{1}{2}n(n-1)\hbar p^{n-2} + \mathrm{i}nqp^{n-1}] \tag{3.8}$$

$$\mathrm{d}(p^n) = \mathrm{d}p\,np^{n-1} + \mathrm{d}q\,c[\tfrac{1}{2}\mathrm{i}n(n-1)qp^{n-2} + \tfrac{1}{6}\hbar n(n-1)(n-2)p^{n-3}]. \tag{3.9}$$

It is sometimes convenient to have (3.1) in the form

$$\Theta(q)^k_\ell = q\,\delta^k_\ell \qquad \Theta(\mathbf{1})^k_\ell = \delta^k_\ell$$
$$\Theta(p)^k_\ell = p\delta^k_\ell + \mathrm{i}cq\delta^k_2\delta^1_\ell.$$

Together with (2.13) this implies

$$\Theta(f)^1_2 = 0 \qquad (\forall f \in \mathcal{A}) \tag{3.10}$$

(since f is a function of p and q) and then also

$$\Theta(f)^1_1 = \Theta(f)^2_2 = f. \tag{3.11}$$

The only complicated term is $\Theta(f)^2_1$ which for $f = \sum f_{mn}q^m p^n$ turns out to be

$$\Theta(f)^2_1 = c\sum f_{mn}q^m n[\tfrac{1}{2}\hbar(n-1)\,p^{n-2} + \mathrm{i}q\,p^{n-1}]. \tag{3.12}$$

According to our next result, the deformation of the differential calculus can be transformed away by a nonlinear transformation.

† Here, $[,]_-$ means commutator and $[,]_+$ anticommutator.

Theorem 3.1. The transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q \\ p + \frac{c}{6\hbar} q^3 \end{pmatrix} = \begin{pmatrix} q' \\ p' \end{pmatrix} \tag{3.13}$$

leaves the canonical commutation relation invariant, preserves Hermiticity, and transforms the differential calculus to the ordinary one.

Proof. The transformation obviously preserves Hermiticity and the canonical commutation relation. A simple calculation shows that in terms of q' and p' we have (3.1) with $c = 0$. □

The transformation $\mathcal{A} \to \mathcal{A}$ given by (3.13) can also be described as

$$f \mapsto U^{-1} f U \qquad (f \in \mathcal{A}) \tag{3.14}$$

with

$$U = \exp\left(\mathrm{i} \frac{c}{24\hbar^2} q^4\right) \tag{3.15}$$

and is therefore invertible.

It is sometimes convenient to regard $\hbar$ as a parameter. The value $\hbar = 0$ then corresponds to classical mechanics where p and q are represented as functions on a phase space. Although in the derivation of the differential calculus (3.1) and (3.2) we relied heavily on the assumption $\hbar \neq 0$, the differential calculus remains consistent in the limit $\hbar \to 0$. The deformation parameter c survives this classical limit. But in contrast to the non-commutative case there is no way to transform it away by a change of the (commutative) coordinates p and q. Whereas different values of the parameter c distinguish different calculi in the classical situation ($\hbar = 0$), we may regard them as equivalent in the non-commutative case ($\hbar \neq 0$)!

So far we have ignored problems arising from the fact that we are dealing with an algebra of unbounded operators. Our formal manipulations do make sense if we restrict the operators p and q (in the Schrödinger representation) to a suitable dense invariant domain (e.g. the Schwartz space $\mathcal{S}(\mathbb{R})$). Adjoint operators are then always understood to be restricted to this domain and one has to make sure that they also leave the common domain invariant. Indeed, the algebra generated by p and q satisfying canonical commutation relations is an example of an O^*-algebra which is a $*$-algebra $\mathcal{A}$ of linear operators defined on a common dense linear subspace of Hilbert space which is invariant under $\mathcal{A}$ [13].

The usual way to circumvent (or rather to hide) the problems with the unboundedness of the operators p and q is to consider only the bounded operators

$$x(s) = \mathrm{e}^{\mathrm{i}sq} \qquad y(t) = \mathrm{e}^{\mathrm{i}tp}. \tag{3.16}$$

A formal† calculation using (2.1) leads to

$$x(s)y(t) = \mathrm{e}^{-\mathrm{i}\hbar st} y(t) x(s). \tag{3.17}$$

† (3.17) and (2.1) are not equivalent. In particular, (3.17) admits finite-dimensional representations in contrast to (2.1) [14].

In principle we could have started with this equation and considered differential calculus on the algebra of bounded operators generated by $x(s)$ and $y(t)$. It is more difficult then, however, to reveal the symplectic structure of quantum mechanics in the sense of section 6. For fixed values of t and s, the Weyl relations (3.17) can be viewed as the commutation relations of a 'quantum plane' [12, 10]. We have $SL(2,\mathbb{R})$ as the group of linear transformations leaving the canonical commutation relation invariant. Similarly, we may ask for linear transformations leaving the Weyl relation invariant. In this case we have non-trivial such transformations only if we allow non-commuting operators instead of numbers as entries of the transformation matrix. This leads to the 'quantum group' $GL_r(2)$ with $r = \exp(-i\hbar st)$.

There is a consistent differential calculus on the quantum plane $xy = r_0 yx$ with the following commutation relations between x, y and their differentials:

$$
\begin{aligned}
x\,\mathrm{d}x &= r_0^2\,\mathrm{d}x\,x \\
x\,\mathrm{d}y &= r_0\,\mathrm{d}y\,x + (r_0^2 - 1)\,\mathrm{d}x\,y \\
y\,\mathrm{d}x &= r_0\,\mathrm{d}x\,y \\
y\,\mathrm{d}y &= r_0^2\,\mathrm{d}y\,y
\end{aligned}
$$

(see [10]). One might expect that this is the 'exponentiated form' of a differential calculus on the algebra $\mathcal{A}$ generated by p and q. Then $x(s)$ and $y(t)$ would play the role of x and y, respectively, and we have to replace r_0 by $r = \exp(-i\hbar st)$. However, the first and the last commutation relations above are inconsistent with r depending on s and t.

Rather, the standard differential calculus on $\mathcal{A}$ leads to

$$
\begin{aligned}
x\,\mathrm{d}x &= \mathrm{d}x\,x \\
x\,\mathrm{d}y &= r\,\mathrm{d}y\,x \\
y\,\mathrm{d}x &= r^{-1}\,\mathrm{d}x\,y \\
y\,\mathrm{d}y &= \mathrm{d}y\,y
\end{aligned}
$$

dropping the dependence on the parameters s and t. More complicated commutation relations are obtained via automorphisms of $\mathcal{A}$ (like (3.13)).

4. Vector fields and flows

In classical mechanics the dynamics of a system is described by a vector field on its phase space. Such a vector field can be regarded as a derivation on the commutative algebra of (smooth) functions of the phase space. In order to generalize the notion of a vector field to non-commutative algebras, one possibility is to define the latter as a derivation of the algebra. This is the point of view taken in [3]. In the framework of non-commutative geometry which we use in this work (and which is also used in the theory of quantum groups) there is another more natural definition of a vector field (see (4.1)). According to that definition, a vector field, in general, does not have the derivation property. On the other hand, derivations play an important role in quantum mechanics. In order to clarify the relation between non-commutative geometry and quantum mechanics we will discuss these facts in this section.

We define vector fields associated with the 'coordinates' ξ^i by [9, 10]†

$$\mathrm{d}f = \mathrm{d}\xi^i \partial_i f \qquad \forall f \in \mathcal{A}. \tag{4.1}$$

The Leibniz rule together with (2.16) then implies

$$\partial_j \xi^i = \delta^i_j + \Theta^{i\ell}_{kj}\, \xi^k\, \partial_\ell. \tag{4.2}$$

For the consistent differential calculus given by (3.1) and (3.2) this leads to the following commutation relations between vector fields and algebra elements:

$$\begin{aligned} \partial_q q &= \mathbf{1} + q\partial_q \\ \partial_q p &= p\partial_q + \mathrm{i}cq\partial_p \\ \partial_p q &= q\partial_p \\ \partial_p p &= \mathbf{1} + p\partial_p. \end{aligned} \tag{4.3}$$

Using $\mathrm{d}^2 = 0$, equation (4.1) implies

$$\mathrm{d}\xi^i\, \mathrm{d}\xi^j\, \partial_j \partial_i = 0. \tag{4.4}$$

Together with (3.2) this leads to the vector field commutation relation

$$[\partial_q, \partial_p] = 0 \tag{4.5}$$

which turns out to be consistent with (4.3). This means that commuting p or q from left to right through (4.5) does not yield additional restrictions on the differential calculus. Whereas ∂_p satisfies the (ordinary) derivation rule

$$\partial_p(fh) = (\partial_p f)h + f\partial_p h \qquad \forall f, h \in \mathcal{A} \tag{4.6}$$

∂_q obeys the rule

$$\partial_q(fh) = (\partial_q f)h + (f\partial_q + \Theta(f)^2_1 \partial_p)h. \tag{4.7}$$

Lemma 4.1. A vector field $X = X^1(q,p)\partial_q + X^2(q,p)\partial_p$ is a derivation if and only if it is a linear combination (with complex coefficients) of the two vector fields ∂_p and

$$\nabla_q := \partial_q - \frac{c}{2\hbar} q^2 \partial_p. \tag{4.8}$$

Proof. If X is a derivation, applying it to simple monomials in q and p and using (4.3) reduces it down to a linear combination of the two vector fields ∇_q and ∂_p. That these vector fields are indeed derivations is most easily seen if we use theorem 3.1, since $\nabla_q = \partial_{q'}$. □

† Occasionally we write ∂_{ξ_i} instead of ∂_i.

Next we would like to generalize the classical notion of the flow of a vector field to our framework. A natural definition appears to be the following. The flow of a vector field X is a family of linear mappings $\varphi_t : \mathcal{A} \to \mathcal{A}$, differentiable with respect to the real parameter t, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t f) = \varphi_t(Xf) \qquad (\forall f \in \mathcal{A}, \forall t \in I) \tag{4.9}$$

and $\varphi_0 = \mathrm{id}$ (the identity mapping on $\mathcal{A}$). I is some interval containing 0. The 'integral curve' of X with initial data $f \in \mathcal{A}$ is then $\varphi_t f$. In order to solve the flow equation one has to choose a basis of $\mathcal{A}$ and, using the linearity of X and φ_t, understand it as an infinite-dimensional matrix equation.

Remark. In order to clarify what we mean by differentiability of a flow we need a topology on the algebra $\mathcal{A}$. An algebra of bounded operators is a Banach space and we have a natural topology induced by the operator norm, the uniform topology. For algebras of unbounded operators there is no natural choice of a topology. A topology on such algebras which can be considered as a generalization of the uniform topology has been introduced and investigated in [15], see also [13, 16]. This can be used to make sense of the left-hand side of (4.9). We are left with the question under which conditions on X does the flow exist, is unique and has the (semi-) group property $\varphi_s \varphi_t = \varphi_{s+t}$.

In (4.9) we should look at the vector field X as a linear operator on the algebra $\mathcal{A}$. In [17] one finds the *flow of a linear operator* on a Banach space with dense domain $\mathcal{D}$ defined as a collection of (linear) bijective maps $\varphi_t : \mathcal{D} \to \mathcal{D}$, differentiable with respect to t, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t f = X \varphi_t f \tag{4.10}$$

and the semi-group property (see also [18]). These conditions imply

$$\varphi_t X = X \varphi_t \tag{4.11}$$

and therefore our flow equation (4.9). □

In quantum mechanics, time evolution is an automorphism of the operator algebra. It is natural to think that this automorphism could be described as the flow of some vector field. However, this does not work because of the following reason. It is easily verified that φ_t is a family of endomorphisms of $\mathcal{A}$, i.e.

$$\varphi_t(fh) = (\varphi_t f)(\varphi_t h) \qquad (\forall f, h \in \mathcal{A}, \forall t \in I) \tag{4.12}$$

if and only if the generating vector field X is a derivation. But lemma 4.1 tells us that there are too few vector fields which are derivations. We may also ask for flows which preserve the Hermiticity of Hermitian initial data. We call a flow 'strongly Hermitian'† if

$$(\varphi_t f)^\dagger = \varphi_t(f^\dagger) \qquad (\forall f \in \mathcal{A}, \forall t \in I). \tag{4.13}$$

† In the language of $*$-algebras this is called an $*$-homomorphism.

For the generating vector field this means that

$$(Xf)^\dagger = X(f^\dagger) \qquad (\forall f \in \mathcal{A}). \tag{4.14}$$

Equation (4.1) which defines the vector fields ∂_i implies

$$\mathrm{d}f = (\mathrm{d}f^\dagger)^\dagger = (\partial_j f^\dagger)^\dagger \mathrm{d}\xi^j = \mathrm{d}\xi^i \Theta((\partial_j f^\dagger)^\dagger)^j_i \tag{4.15}$$

and therefore

$$\partial_i f = \Theta((\partial_j f^\dagger)^\dagger)^j_i . \tag{4.16}$$

Using (3.10) and (3.11) the last equation implies

$$(\partial_q f)^\dagger = \partial_q(f^\dagger) + \Theta((\partial_p f^\dagger)^\dagger)^{2\dagger}_1 \tag{4.17}$$

$$(\partial_p f)^\dagger = \partial_p(f^\dagger). \tag{4.18}$$

Hence, ∂_p is strongly Hermitian. ∂_q is strongly Hermitian if and only if the parameter c vanishes. Some more analysis reveals that the only strongly Hermitian vector fields are real linear combinations of the two vector fields in lemma 4.1. Time evolution in quantum mechanics preserves Hermiticity. Again, these results teach us that it is not possible to describe time evolution as the flow of a vector field.

Nevertheless, there is a way to recover quantum mechanics in our geometrical framework and there is a sufficiently large class of vector fields which we may still think of as generators of time evolution, but necessarily in a weaker sense as attempted above. With a vector field X we associate a family of *endomorphisms* of $\mathcal{A}$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_t q = \hat{\varphi}_t(Xq) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_t p = \hat{\varphi}_t(Xp) \qquad \hat{\varphi}_0 = \mathrm{id}. \tag{4.19}$$

In order for the endomorphism $\hat{\varphi}_t$ of an 'evolution vector field' X to describe quantum mechanical time evolution, it has to preserve the canonical commutation relations, i.e.

$$[q(t), p(t)] := [\hat{\varphi}_t q, \hat{\varphi}_t p] = [q, p] = \mathrm{i}\hbar\mathbb{1} \tag{4.20}$$

as well as the Hermiticity of p and q. In terms of the vector field X, these conditions read

$$[Xq, p] = [Xp, q] \tag{4.21}$$

and

$$(Xq)^\dagger = Xq \qquad (Xp)^\dagger = Xp \tag{4.22}$$

respectively,

$$[X^1, p] = [X^2, q] \qquad (X^1)^\dagger = X^1 \qquad (X^2)^\dagger = X^2 \tag{4.23}$$

if we express X as $X = X^1\partial_q + X^2\partial_p$.

We will call a vector field X 'Hermitian' if it satisfies (4.22), i.e. we require (4.14) only for p and q.

As an intermediate step towards (4.19) we could have associated with each vector field X a derivation $\hat{X}$ (which is not a vector field, in general) such that

$$\hat{X}q = Xq \qquad \hat{X}p = Xp. \tag{4.24}$$

The derivation $\hat{X}$ then generates $\hat{\varphi}_t$ which will be a (semi-) group of endomorphisms of $\mathcal{A}$, under suitable technical conditions.

5. Inner product of vector fields and differential forms

In classical differential geometry one has an inner product $\lrcorner$ between vector fields and differential forms. In particular, it is needed in classical mechanics to associate a 'Hamiltonian vector field' with a function on phase space. The aim of this section is to generalize the inner product to the non-commutative algebra $\mathcal{A}$.

In the previous section we have introduced vector fields ∂_i by (4.1). With $f = \xi^j$ this equation implies

$$\delta_i^j = \partial_i\xi^j = \partial_i\lrcorner \mathrm{d}\xi^j \tag{5.1}$$

where the last equality defines an inner product between vector fields and 1-forms. If $\rho = \mathrm{d}\xi^i\rho_i$ and $X = X^j\partial_j$, then

$$X\lrcorner\rho = X^i\rho_i. \tag{5.2}$$

We extend the inner product to higher order differential forms via the operator equation

$$\partial_i\lrcorner\mathrm{d}\xi^j = \delta_i^j - \mathrm{d}\xi^k Q_{ki}^{j\ell}\partial_\ell\lrcorner \tag{5.3}$$

where $Q_{ki}^{j\ell} \in \mathcal{A}$. For $Q_{ki}^{j\ell} = \delta_k^j\delta_i^\ell$ we recover the classical rule. Q has to satisfy certain consistency conditions. These arise by acting with $\lrcorner$ on differential form expressions which have to vanish identically. First we must have

$$\partial_i\lrcorner[(1+P)\,\mathrm{d}\xi\,\mathrm{d}\xi] = 0 \tag{5.4}$$

as a consequence of the commutation relations (2.51) which the differentials have to obey. This amounts to

$$(1+P)(Q-1) = 0 \tag{5.5}$$

and therefore

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\alpha & 1-\beta & -\delta & -\gamma \\ \alpha & \beta & 1+\delta & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

with $\alpha, \beta, \gamma, \delta \in \mathcal{A}$. (5.4) can now be written as

$$\begin{aligned}
\partial_q \lrcorner \mathrm{d}q &= 1 - \mathrm{d}q\, \partial_q \lrcorner + (\mathrm{d}q\, \alpha + \mathrm{d}p\, \delta)\, \partial_p \lrcorner \\
\partial_q \lrcorner \mathrm{d}p &= -[\mathrm{d}q\, \alpha + \mathrm{d}p(1+\delta)]\partial_q \lrcorner \\
\partial_p \lrcorner \mathrm{d}q &= [-\mathrm{d}q(1-\beta) + \mathrm{d}p\, \gamma]\partial_p \lrcorner \\
\partial_p \lrcorner \mathrm{d}p &= 1 - (\mathrm{d}q\, \beta + \mathrm{d}p\, \gamma)\partial_q \lrcorner - \mathrm{d}p\, \partial_p \lrcorner .
\end{aligned} \tag{5.7}$$

As a further consistency condition we have

$$\partial_i \lrcorner (\mathrm{d}\xi^k\, \mathrm{d}\xi^\ell\, \mathrm{d}\xi^m) = 0 \tag{5.8}$$

since triple products of differential forms vanish identically as a consequence of the commutation relations (2.51), respectively (3.2). Together with (5.4) this means

$$\delta_i^k\, \mathrm{d}\xi^\ell\, \mathrm{d}\xi^m - \mathrm{d}\xi^n Q^{k\ell}_{ni}\, \mathrm{d}\xi^m + \mathrm{d}\xi^n Q^{kt}_{ni}\, \mathrm{d}\xi^s Q^{\ell m}_{st} = 0 . \tag{5.9}$$

In the following we will assume the entries of Q to be complex numbers (instead of more general elements of $\mathcal{A}$). The last equation then becomes

$$(1 - \hat{Q}_{12} + \hat{Q}_{23}\hat{Q}_{12})(1 - P_{23}) = 0 \tag{5.10}$$

(where $\hat{Q} = QP$) and imposes the condition

$$\alpha\gamma = \beta\delta \tag{5.11}$$

between them.

As a further condition to narrow down the freedom in the inner product, we may require

$$\mathrm{d}\xi^i (\partial_i \lrcorner \rho) = r\rho \tag{5.12}$$

for any r-form ρ, a familiar formula in classical differential calculus†. This means

$$(1 - Q)(1 - P) = 0 \tag{5.13}$$

and therefore

$$\delta = \beta . \tag{5.14}$$

† The equation (4.1) which defines the vector fields ∂_i can also be written in the form $\mathrm{d}f = \mathrm{d}\xi^i\, \partial_i \lrcorner \mathrm{d}f$. (5.12) generalizes this formula to r-forms.

6. Non-commutative symplectic geometry

Classical mechanics deals with the commutative algebra of functions on a phase space. The structure of classical mechanics is most elegantly described in terms of symplectic geometry. Quantum mechanics works with the non-commutative Heisenberg algebra $\mathcal{A}$. After the preparations in the previous sections we can now address the question whether quantum mechanics can be understood as non-commutative symplectic geometry.

The analogue of the canonical symplectic form of classical mechanics is†

$$\omega = \mathrm{d}p \wedge \mathrm{d}q. \tag{6.1}$$

ω is hermitian‡, i.e. $\omega^\dagger = \omega$. We call a transformation φ of p and q 'canonical' if it leaves ω invariant, i.e.

$$\mathrm{d}\varphi(p) \wedge \mathrm{d}\varphi(q) = \mathrm{d}p \wedge \mathrm{d}q. \tag{6.2}$$

A vector field X is called 'canonical' if the associated endomorphism $\hat{\varphi}_t$ of $\mathcal{A}$ (see section 4) is canonical. We then have the following result.

Lemma 6.1. A vector field X is canonical iff

$$\partial_q(Xq) + \partial_p(Xp) = 0. \tag{6.3}$$

An evolution vector field X is canonical iff

$$\partial_q(Xq) = \frac{1}{\mathrm{i}\hbar}[Xq, p]. \tag{6.4}$$

Proof. Using (3.2) and the fact that $\mathrm{d}q$ commutes with all algebra elements (see (3.1)), we find

$$\begin{aligned}
\mathrm{d}p \wedge \mathrm{d}q[\partial_q(Xq) + \partial_p(Xp)] &= \mathrm{d}p \wedge \mathrm{d}q\, \partial_q(Xq) - \mathrm{d}q \wedge \mathrm{d}p\, \partial_p(Xp) \\
&= \mathrm{d}p \wedge \mathrm{d}(Xq) - \mathrm{d}q \wedge \mathrm{d}(Xp) \\
&= \mathrm{d}(Xp) \wedge \mathrm{d}q + \mathrm{d}p \wedge \mathrm{d}(Xq) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} [\mathrm{d}p(t) \wedge \mathrm{d}q(t)]
\end{aligned}$$

where $q(t) = \hat{\varphi}_t q$ and $p(t) = \hat{\varphi}_t p$. This vanishes iff§

$$\mathrm{d}p(t) \wedge \mathrm{d}q(t) = \mathrm{d}p \wedge \mathrm{d}q$$

which is the statement that $\hat{\varphi}_t$ is canonical. ∂_p is a derivation and we can express it as a commutator:

$$\partial_p = \frac{1}{\mathrm{i}\hbar}[q, \cdot].$$

† Because of the fact that the differentials of p and q satisfy the usual anticommutation rule there is no operator ordering ambiguity here.

‡ If we use Connes' conventions for †, ω is anti-Hermitian: $\omega^\dagger = \mathrm{d}(q^\dagger)\mathrm{d}(p^\dagger) = \mathrm{d}q \wedge \mathrm{d}p = -\omega$.

§ Here we assume that (4.19) has a unique solution.

If X is an evolution vector field, then

$$[Xq, p] = [Xp, q] = -\mathrm{i}\hbar\partial_p(Xp).$$

This shows that (6.3) and (6.4) are equivalent for an evolution vector field. □

The $\mathrm{SL}(2,\mathbb{R})$ transformations used in section 2 to classify differential calculi and also the nonlinear transformation (3.13) which 'undeforms' our class of consistent differential calculi are 'canonical' transformations. This suggests that it is sufficient to formulate symplectic geometry with the undeformed differential calculus, i.e. (3.1) with $c = 0$ (see, however, the discussion at the end of this section). In this case an evolution vector field is automatically canonical (since ∂_q is then a derivation and (6.4) becomes an identity).

With $f \in \mathcal{A}$ we associate a vector field X_f via

$$X_f \lrcorner \omega = -\mathrm{d}f \tag{6.5}$$

copying the classical definition of Hamiltonian vector fields. Writing

$$X_f = X_f^1 \partial_q + X_f^2 \partial_p \tag{6.6}$$

$$\mathrm{d}f = f_1\,\mathrm{d}q + f_2\,\mathrm{d}p \tag{6.7}$$

and using

$$\partial_q \lrcorner \omega = -\alpha\,\mathrm{d}q - (1+\delta)\,\mathrm{d}p \tag{6.8}$$

$$\partial_p \lrcorner \omega = (1-\beta)\,\mathrm{d}q - \gamma\,\mathrm{d}p \tag{6.9}$$

(6.5) becomes

$$\begin{pmatrix} \alpha & \beta - 1 \\ 1+\delta & \gamma \end{pmatrix} \begin{pmatrix} X_f^1 \\ X_f^2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{6.10}$$

In order to be able to assign a vector field to any $f \in \mathcal{A}$ we have to require that

$$\alpha\gamma + (1-\beta)(1+\delta) \neq 0 \tag{6.11}$$

or, using (5.11),

$$\mathcal{D} := 1 + \delta - \beta \neq 0 \tag{6.12}$$

which is always satisfied if we accept the additional condition (5.12). (6.12) allows us to invert (6.10):

$$\begin{pmatrix} X_f^1 \\ X_f^2 \end{pmatrix} = \frac{1}{\mathcal{D}} \begin{pmatrix} \gamma & 1-\beta \\ -1-\delta & \alpha \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{6.13}$$

Then, in particular,

$$X_q = \frac{1}{\mathcal{D}}[\gamma \partial_q - (1+\delta)\,\partial_p] \tag{6.14}$$

$$X_p = \frac{1}{\mathcal{D}}[(1-\beta)\partial_q + \alpha \partial_p]. \tag{6.15}$$

Furthermore, for the Hamiltonian

$$H = \frac{1}{2\,m}p^2 + V(q) \tag{6.16}$$

we have

$$\mathrm{d}H = V'(q)\mathrm{d}q + \frac{1}{m}p\,\mathrm{d}p \tag{6.17}$$

and therefore

$$X_H = \frac{1}{\mathcal{D}}\left[\gamma V'(q) + \frac{1-\beta}{m}p\right]\partial_q + \frac{1}{\mathcal{D}}\left[\frac{\alpha}{m}p - (1+\delta)V'(q)\right]\partial_p. \tag{6.18}$$

X_H should determine dynamics. It therefore should be a canonical evolution vector field (for a sufficiently large class of potentials). This means that X_H has to satisfy (4.23) which requires $\alpha = \beta = \gamma = \delta = 0$ (if we assume (5.12), otherwise a real β or a real δ would survive). We then have the laws of the classical differential calculus. The endomorphism $\hat{\varphi}_t$ generated by X_H is now determined by

$$\dot{q}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_t q = \hat{\varphi}_t(X_H q) = \frac{1}{m}p(t) \tag{6.19}$$

$$\dot{p}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\varphi}_t p = \hat{\varphi}_t(X_H p) = -V'(q(t)) \tag{6.20}$$

which are the Heisenberg equations of motion of quantum mechanics if we represent p and q by operators on some Hilbert space.

The supplementary conditions which we had to impose on the non-commutative symplectic geometry in order to recover the equations of motion of quantum mechanics amount to the requirement that the derivation $\hat{X}_H$ associated with the Hamiltonian vector field X_H is given by

$$\hat{X}_H = \frac{\mathrm{i}}{\hbar}\mathrm{ad}(H) := \frac{\mathrm{i}}{\hbar}[H, \cdot]. \tag{6.21}$$

The above results were obtained by using the possibility undeforming the differential calculus by canonical transformations. But such a transformation acts non-trivially on a Hamiltonian. *In particular, this shows that dynamics can be shifted from the* Hamiltonian to the differential calculus and *vice versa*! Physics can therefore be described in different ways. One extreme is the point of view taken above where the differential calculus is undeformed. Another extreme is to choose the free Hamiltonian (i.e. only the kinetic energy) and represent the interaction as a deformation of the differential calculus†. The latter point of view can be regarded as a kind of geometrization of interaction. Again, we have to stress that all this only works if $\hbar$ (regarded as a parameter) does not vanish.

† In this case one has to find a suitable formula to replace (6.5).

7. Conclusions

In classical geometry we are already used to dealing with a non-commutative algebra, namely the algebra of differential forms. Physically we may think of it as a mathematical formulation of our classical conception of how to measure volumes. In this sense the 'standard' differential calculus is distinguished among the many possible consistent differential calculi which exist on a commutative algebra, in particular on the algebra of functions on a phase space. So far there seems to be no argument why one should consider a 'deformed' differential calculus in classical physics.

A 'quantum group' is a (special kind of) non-commutative Hopf algebra. A differential calculus on a matrix quantum group (e.g. a deformation of $SU(2)$) allows one to define a 'quantum Lie algebra', i.e. generators of the matrix quantum group [9]. In general, standard differential calculus is not consistent with the algebra structure. In this case one is forced to choose a 'deformed' calculus. The non-commutativity of the algebra severely constrains the possibilities of consistent differential calculi.

The algebra $\mathcal{A}$ generated by operators p and q satisfying the canonical commutation relation is consistent with ordinary differential calculus, but also allows certain deformations. Only with a certain linearity assumption (see (2.15)) we were able to solve the consistency conditions completely. It then turned out that the allowed deformations could be transformed away by linear ($SL(2,\mathbb{R})$) transformations and a nonlinear transformation of p and q. These transformations preserve the canonical commutation relation and Hermiticity. This result has no counterpart in the case of a commutative algebra $\mathcal{A}$. On the other hand, it shows that our linearity assumption should be relaxed because nonlinear transformations of p and q will map a linear Θ into a nonlinear one. This raises the question as to whether a nonlinear Θ can be transformed into a linear one by such a transformation (which preserves the canonical commutation relation and the Hermiticity of p and q). We do not know the answer yet.

The fact that we were able to transform all the differential calculi which we found to the standard one made it easy to take definitions from classical symplectic geometry to the non-commutative geometry. This concerns in particular the definition of Hamiltonian vector fields. If there are deformations of the differential calculus, we have (additional) operator ordering ambiguities here and (6.5) is no longer the correct formula. Some more work is needed to clarify what the correct generalization is. In particular, this is important for developing the idea to represent interactions 'geometrically' as deformations of the differential calculus.

We have discussed in which sense the Heisenberg picture of quantum mechanics can be understood as (non-commutative) symplectic geometry. The correspondence is not perfect since quantum mechanical time evolution of some observable $f(p,q)$ does not correspond to the action on f of the flow φ_t generated by a Hamiltonian vector field. It is rather given by $\hat{\varphi}_t f = f(\hat{\varphi}_t p, \hat{\varphi}_t q)$ where $\hat{\varphi}_t$ is an endomorphism of the operator algebra, associated with the vector field.

Our restriction to a single pair of 'position' and 'momentum' operators is not essential. The problem to find the most general consistent differential calculus increases, however, with the complexity of the algebra.

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Appendix. Proof of lemma 2.1

We will need the following two lemmata.

Lemma A.1.

$$A\xi\xi = 0 \Leftrightarrow A = 0 \qquad (\text{if } \hbar \neq 0). \tag{A.1}$$

Proof.

$$\begin{aligned} 0 = A\xi\xi &= \tfrac{1}{2}[A(\mathbf{1}+P) + A(\mathbf{1}-P)]\xi\xi \\ &= \tfrac{1}{2}[A(\mathbf{1}+P)\xi\xi + \mathrm{i}\hbar A\epsilon]. \end{aligned}$$

For $\hbar \neq 0$ this implies $A = 0$. □

If X and Y are 2×2 matrices, we define (cf section 2)

$$\mathrm{Tr}_1(X \otimes Y) = (\mathrm{Tr}\, X)Y. \tag{A.2}$$

Tr_1 has the following properties.

Lemma A.2.

$$\mathrm{Tr}_1(P(X \otimes Y)) = X \cdot Y \qquad \mathrm{Tr}_1((X \otimes Y)P) = Y \cdot X. \tag{A.3}$$

Proof.

$$\mathrm{Tr}_1(P(X \otimes Y))^i_j = (P(X \otimes Y))^{ki}_{kj} = P^{ki}_{\ell m} X^\ell_k Y^m_j = X^i_k Y^k_j.$$

The second formula can be checked in the same way. □

With these preparations we now proceed to the proof of the asserted formulae. Using the Yang–Baxter equation†

$$P_{12}P_{23}P_{12} = P_{23}P_{12}P_{23} \tag{A.4}$$

we find

$$\begin{aligned} (\mathbf{1}-P)_{12}P_{23}P_{12}\mathrm{d}\xi\,\xi\,\xi &= P_{23}P_{12}(\mathbf{1}-P)_{23}\mathrm{d}\xi\,\xi\xi \\ &= \epsilon_{ij}\xi^i\xi^j P_{23}P_{12}\mathrm{d}\xi\epsilon \\ &= \epsilon_{ij}\xi^i\xi^j\epsilon\mathrm{d}\xi \\ &= (\mathbf{1}-P)_{12}\xi\,\xi\,\mathrm{d}\xi \\ &= (\mathbf{1}-P)_{12}\hat{\Theta}_{23}\xi\,\mathrm{d}\xi\,\xi \\ &= (\mathbf{1}-P)_{12}\hat{\Theta}_{23}\hat{\Theta}_{12}\mathrm{d}\xi\,\xi\xi. \end{aligned}$$

† This acts on a threefold tensor product and the indices refer to the respective components.

By lemma A.1 this implies

$$\begin{aligned} 0 &= (\mathbf{1} - P)_{12}(\hat{\Theta}_{23}\hat{\Theta}_{12} - P_{23}P_{12}) \\ &= (\mathbf{1} - P)_{12}(\Theta_{23}\Theta_{13} - \mathbf{1})P_{23}P_{12} \end{aligned}$$

and thus

$$(\mathbf{1} - P)_{12}(\Theta_{23}\Theta_{13} - \mathbf{1}) = 0.$$

With (2.23) the last equation can be converted into

$$\begin{aligned} 0 &= (\mathbf{1} - P)_{12}(\Theta_{23} - \bar{\Theta}_{13}) \\ &= (\mathbf{1} - P)_{12}(\Theta_{23} - P_{12}\bar{\Theta}_{23}P_{12}) \\ &= (\mathbf{1} - P)_{12}(\Theta_{23} + \bar{\Theta}_{23}P_{12}) \end{aligned} \tag{A.5}$$

and therefore

$$(\mathbf{1} - P)_{12}(\Theta_{23} + \bar{\Theta}_{23})(\mathbf{1} + P)_{12} = 0.$$

Inserting the decomposition

$$\Theta_{23} + \bar{\Theta}_{23} = \sum_a X_a \otimes Y_a$$

gives

$$0 = \sum_a [(\mathbf{1} - P) \otimes \mathbf{1}][\mathbf{1} \otimes X_a \otimes Y_a][(\mathbf{1} + P) \otimes \mathbf{1}].$$

We may choose the matrices Y_a to be linearly independent. Then

$$\begin{aligned} 0 &\doteq (\mathbf{1} - P)(\mathbf{1} \otimes X_a)(\mathbf{1} + P) \\ &= \mathbf{1} \otimes X_a + (\mathbf{1} \otimes X_a)P - P(\mathbf{1} \otimes X_a) - X_a \otimes \mathbf{1}. \end{aligned}$$

Acting with Tr_1 on this equation and using lemma A.2 yields

$$X_a = \tfrac{1}{2}(\mathrm{Tr}\, X_a)\mathbf{1}$$

so that

$$\Theta + \bar{\Theta} = \sum_a \tfrac{1}{2}(\mathrm{Tr}\, X_a)\mathbf{1} \otimes Y_a.$$

Now we apply $P - \mathbf{1}$ from the left and P from the right. Using (2.22) we find

$$\begin{aligned} 2(\mathbf{1} - P) &= \sum_a \tfrac{1}{2}(\mathrm{Tr}\, X_a)(P - \mathbf{1})(\mathbf{1} \otimes Y_a)P \\ &= \sum_a \tfrac{1}{2}(\mathrm{Tr}\, X_a)[\mathbf{1} \otimes Y_a - (\mathbf{1} \otimes Y_a)P]. \end{aligned}$$

Evaluation of this equation with Tr_1 yields

$$2 \cdot \mathbf{1} = \tfrac{1}{2} \sum_a (\mathrm{Tr}\, X_a) Y_a$$

and leads to

$$\Theta + \bar{\Theta} = 2 \cdot \mathbf{1}. \tag{A.6}$$

From (A.5) we also have

$$(\mathbf{1} - P)_{12} (\Theta_{23} - \bar{\Theta}_{23})(\mathbf{1} - P)_{12} = 0$$

and, using (A.6),

$$(\mathbf{1} - P)_{12}(\Theta - \mathbf{1})_{23}(\mathbf{1} - P)_{12} = 0.$$

Inserting the decomposition

$$\Theta - \mathbf{1} = \sum_a Z_a \otimes Y_a$$

we find

$$\mathbf{1} \otimes Z_a + Z_a \otimes \mathbf{1} - P(\mathbf{1} \otimes Z_a) - (\mathbf{1} \otimes Z_a) P = 0.$$

Application of Tr_1 yields

$$\mathrm{Tr}\, Z_a = 0$$

and thus

$$\mathrm{Tr}_1(\Theta - \mathbf{1}) = 0. \tag{A.7}$$

(A.6) and (A.7) are the asserted formulae.

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